Trial Higher School Certificate 2011
Extension 2

Time: 3 hours **Total 120 marks**

Question 1 (15 marks) (begin on a new page)

Marks

a) Find $\int \frac{x}{\sqrt{16-x^2}}\,dx$. **2**

b) By completing the square, find

$$\int \frac{8}{x^2+4x+13}\,dx.$$ **2**

c) i) Find real numbers a, b, c, such that

$$\frac{7x+4}{\left(x^2+1\right)\left(x+2\right)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2} \quad .$$ **3**

ii) Hence, find

$$\int \frac{7x+4}{\left(x^2+1\right)\left(x+2\right)}\,dx \quad .$$ **2**

d) Use integration by parts to find

$$\int x^2 \log_e x\,dx \quad .$$ **3**

e) Use the substitution $u = \cos x$, to find

$$\int \cos^2 x \sin^7 x\,dx.$$ **3**

Question 2 (15 marks) (begin on a new page) **Marks**

a) Let $z = 2 + i,\ w = 1 - i$. Find, in the form $x + iy$,

i) $3z + iw,$ **1**

ii) $z\overline{w},$ **1**

iii) $\frac{5}{z}.$ **1**

b) Let $\alpha = -\sqrt{3} + i.$

i) Express α in modulus - argument form. **2**

ii) Express α^4 in modulus - argument form. **1**

iii) Hence express α^4 in form $x + iy$. **1**

c) Sketch the region in the complex plane where the inequalities

$|z - i| \le 2$ and $0 \le \arg(z - 1) \le \frac{3\pi}{4}$ hold simultaneously. **3**

d) Let $z_1 = 4 + i$ and $z_2 = 1 + 2i.$ Let points *A, B, C* represent the complex numbers $z_1,\ z_2,\ z_1 - z_2$, respectively, on the complex plane.

i) On a diagram, show the points *A, B, C.* Indicate any geometrical relationships on your diagram. **1**

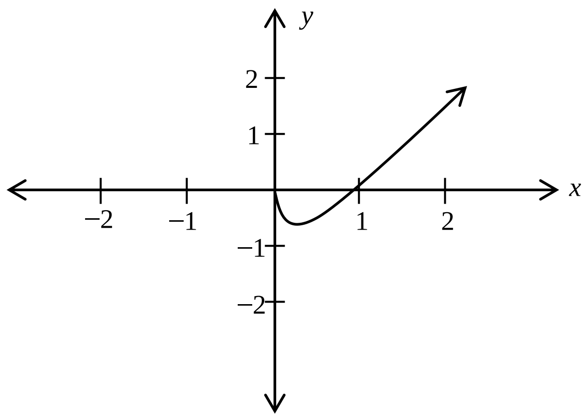
ii) The point *A* is rotated through 90° in the anticlockwise direction about *B* to the point *D*. Write down the complex number represented by *D.* **1**

e) On an Argand diagram, sketch and describe geometrically the locus of *z* such that

$|z| = |z - 4|$. **3**

Question 3 (15 marks) (begin on a new page) **Marks**

a) The diagram below shows the graph of the function $y = f(x)$.



Draw separate, one-third page sketches of the graphs of the following:

i) $y = -f(x)$, **1**

ii) $y = |f(x)|$, **1**

iii) $y = \dfrac{1}{f(x)}$, **2**

iv) $y = [f(x)]^2$. **2**

b) i) On the same set of axes, sketch the graphs of $y = \log_e x$ and $y = \dfrac{2}{x}$. **1**

ii) Hence, on a separate diagram, sketch the graph of $y = \dfrac{2\log_e x}{x}$. **4**

Indicate on your graph any asymptotes and the co-ordinates of any stationary points.

c) Sketch the graph of $y^2 = (x-2)(x-3)$, including any asymptotes. **4**

Question 4 (15 marks) (begin on a new page) **Marks**

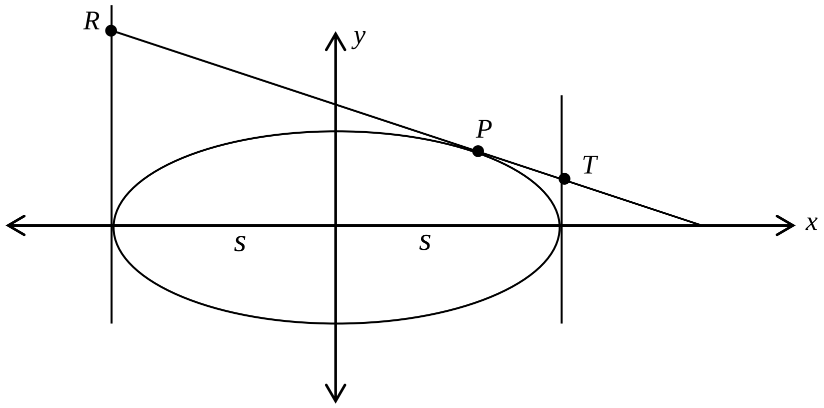
a) Consider the hyperbola with the equation $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

i) What is the eccentricity of the hyperbola? **1**

ii) Find the co-ordinates of the foci and x intercepts of the hyperbola. **2**

iii) Find the equations of the directrices and the asymptotes for the hyperbola. **2**

b)



The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a > b > 0, as shown in the diagram above. The points S and S' are the foci. The tangent at P meets the tangents at the ends of the major axis at R and T.

i) Show that the equation of the tangent at P is given by

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$ **2**

ii) Show that RT subtends a right angle at S. **3**

iii) Show that the points R, T, S, S' are concyclic. **1**

c) Let $Q(x_0, y_0)$ be an external point of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (That is $\frac{x_0^{\ 2}}{a^2} + \frac{y_0^{\ 2}}{b^2} > 1$).

Show that the equation of the chord of contact of the tangents from the point Q to the ellipse, is given by the equation $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$. **4**

Question 5 (15 marks) (begin on a new page) **Marks**

a) When $P(x) = x^4 + ax^2 + bx$ is divided by $x^2 + 1$, the remainder is $x + 2$.
Find the values of a and b, given that these values are real. **3**

b) The graph of $y = 2x^3 - 3x^2 - 12x + k$ has turning points at $x = 2$ and $x = -1$.
Find the values of k such that the equation $y = 2x^3 - 3x^2 - 12x + k = 0$ has three real and distinct roots. **2**

c) A nine-member fund raising committee consists of four students, three teachers and two parents. The committee meets around a circular table, such that all the students sit together as a group, all the teachers sit together as another group, but no teacher sits next to a student.

i) How many different arrangements are possible for the members of the committee to sit around the table. **2**

ii) One student and one parent are related. Given that all arrangements in b i) are equally likely, what is the probability that these two members sit next to each other? **2**

d) i) Write down the three relations which hold between roots α, β, γ of the equation $ax^3 + bx^2 + cx + d = 0$, $a \neq 0$ and the coefficients a, b, c, d. **1**

ii) Consider the equation $36x^3 - 12x^2 - 11x + 2 = 0$.
You are given that the roots α, β, γ of this equation satisfy $\alpha = \beta + \gamma$
Use part i) to find α. **2**

iii) Suppose the equation $x^3 + px^2 + qx + r = 0$ has roots α, β, γ, which satisfy $\alpha = \beta + \gamma$.
Show that $p^3 - 4pq + 8r = 0$. **3**

Question 6 (15 marks) (begin on a new page) **Marks**

a)

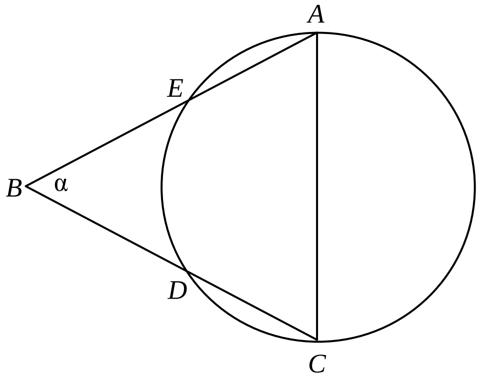


Diagram not to scale

In the diagram above, AC is a diameter of a circle with the point B outside the circle. The intervals BC and BA meet the circle in the points D and E respectively.

Also, $AC = BC$. Let $BA = x$, $BC = y$ and $\angle ABC = \alpha$.

i) Show that $\cos\alpha = \dfrac{x}{2y}$. **2**

ii) Find the length DC in terms of x and y. **4**

b) Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \cos^n t\ dt$, where $n \geq 0$ is an integer.

i) Show that $I_n = \left(\dfrac{n-1}{n}\right) I_{n-2}$ for $n \geq 2$. **2**

ii) Hence find the exact value of I_4. **2**

Question 6 continues on the next page.

6c) i) Let $k > 0$ be a real number. **Marks**

If $\frac{(k+1)^{k-1}}{k^k} < 1$, show that $(k+1)^{k+1} > \frac{(k+1)^{2k}}{k^k}$. **2**

ii) Prove, by mathematical induction, for all integers $n \geq 2$ that

$n^n > (n+1)^{n-1}$. **3**

Question 7 (15 marks) (begin on a new page)

a) Let $I_n = \int \frac{dx}{(x^2+1)^n}$ where $n \geq 1$, is an integer.

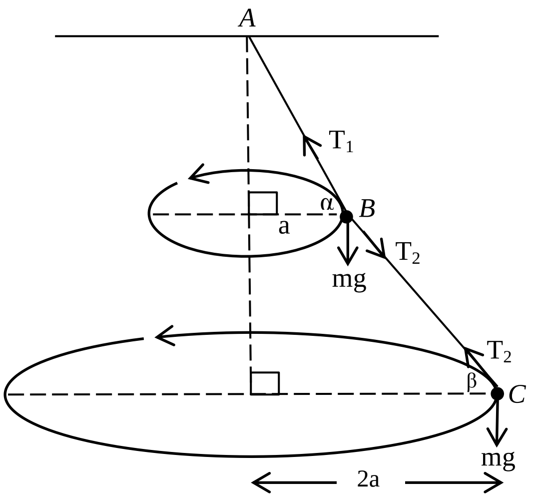
i) Show that, for $n \geq 2$, $I_n = \frac{1}{2(n-1)}\left[\frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}\right]$. **4**

ii) Hence evaluate

$$\int_0^1 \frac{dx}{(x^2+1)^2}.$$ **2**

Question 7 continues on the next page.

7b)



Diagram

not to

scale

A light, inextensible string ABC where $AB = \frac{5a}{3}$ and is inclined at an angle of α to the horizontal, while $BC = \frac{5a}{4}$ and is inclined at an angle β to the horizontal.
At B is attached a particle of mass $7m$ and at C is attached a particle of m.
The end A is attached to a fixed point and the whole system rotates steadily with uniform angular velocity about the vertical through A in such a way that B and C describe horizontal circles of radii a and $2a$ respectively.
The tensions in the strings AB and BC are T_1 and T_2 respectively. The strings remain taught.
The acceleration due to gravity is g.

i) Show that $T_2 = \frac{5}{3}mg$. **3**

ii) Find T_1. **2**

iii) If v_1 is the speed of B and v_2 is the speed of C, find the value of $\frac{v_1}{v_2}$. **4**

Question 8 (15 marks) (begin on a new page) **Marks**

a) The region between the curve $y = \sin x$ and the line $y = 1$, from $x = 0$ to $x = \frac{\pi}{2}$, is rotated around the line $y = 1$.
Using a slicing technique, find the exact volume formed. **4**

b) i) Differentiate with respect to *x*, the function, $h(x)$ given by

$$h(x) = \frac{\log_e x}{x} \quad \text{for } x > 0.$$ **1**

ii) Given that the only stationary point of $y = h(x)$ is a maximum turning point, deduce, without calculating any numerical values, that $e^{\pi} > \pi^{e}$.
$\left[\text{you may assume that } \pi > e\right]$ **2**

c) The diagram shows a boat showroom built on level ground. The length of showroom is 100 m. At one end of the showroom, the shape is a square measuring 20 m by 20 m and at the other end an isosceles triangle of height 20 m and base 10 m. The trapezium *ABCD* is a cross section of the showroom taken parallel to the ends.

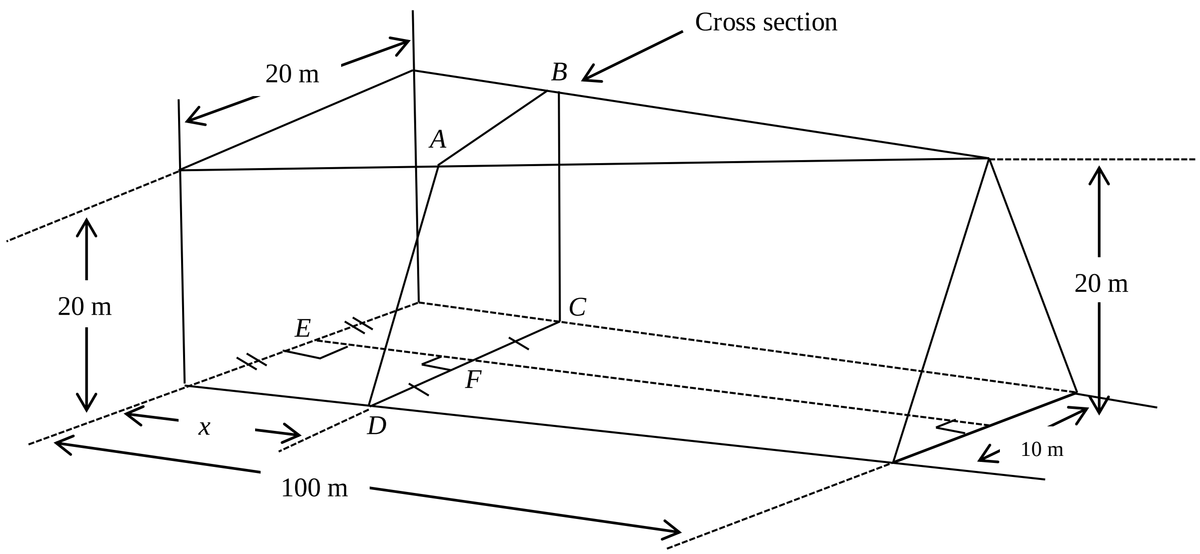


Diagram not to scale.

i) If *EF* is *x* metres in length, show that the length of *DC* is $\left(20 - \frac{x}{10}\right)$ metres. **2**

ii) By considering trapezoidal slices parallel to the ends of the showroom, find the volume enclosed by the showroom. **6**

END OF EXAMINATION

Question 1

(a)(i) Let $u = 16 - x^2$ then $\frac{du}{dx} = -2x$

$$xdx = -\frac{1}{2}du$$

$$\int \frac{x}{\sqrt{16-x^2}}dx = -\frac{1}{2}\int u^{-\frac{1}{2}}du$$
$$= -\frac{1}{2} \times 2u^{\frac{1}{2}}$$
$$= -\sqrt{16-x^2}$$

(b)
$$\int \frac{8}{x^2+4x+13}dx = 8\int \frac{dx}{(x+2)^2+3^2}$$
$$= \frac{8}{3}\tan^{-1}\frac{(x+2)}{3} + c$$

(c)(i)
$$\frac{7x+4}{(x^2+1)(x+2)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2}$$

$7x+4 = (ax+b)(x+2) + c(x^2+1)$

Let $x = -2$ and $x = 0$

$-10 = 5c$ $\quad$ $4 = b(0+2) - 2(0^2+1)$

$c = -2$ $\quad$ $b = 3$

Equating the coefficients of x^2, $0 = a - 2$

$a = 2$

$\therefore a = 2,\ b = 3$ and $c = -2$

(c)(ii)
$$\int \frac{7x+4}{(x^2+1)(x+2)}dx = \int \left(\frac{2x+3}{(x^2+1)} - \frac{2}{(x+2)}\right)dx$$
$$= \int \left(\frac{2x}{(x^2+1)} + \frac{3}{(x^2+1)} - \frac{2}{(x+2)}\right)dx$$
$$= \ln(x^2+1) + 3\tan^{-1}x - 2\ln|x+2| + c$$

Q1

d) $\int x^2 \log_e x \, dx = \log_e x \times \frac{x^3}{3} - \int \frac{x^3}{3}\frac{1}{x} dx$

$= \frac{x^3 \log_e x}{3} - \frac{1}{3}\int x^2 dx$

$= \frac{x^3 \log_e x}{3} - \frac{x^3}{9} + c$

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e) Let $u = \cos x$ then $-du = \sin x dx$

Now $\sin^6 x = (\sin^2 x)^3$

$= (1-\cos^2 x)^3$

$= 1 - 3\cos^2 x + 3\cos^4 x - \cos^6 x$

$\int \cos^2 x \sin^7 x dx = \int \cos^2 x \sin^6 x \sin x dx$

$= \int u^2(1-3u^2+3u^4-u^6)(-du)$

$= \int -u^2 + 3u^4 - 3u^6 + u^8 du$

$= -\frac{u^3}{3} + \frac{3u^5}{5} - \frac{3u^7}{7} + \frac{u^9}{9} + c$

$= -\frac{\cos^3 x}{3} + \frac{3\cos^5 x}{5} - \frac{3\cos^7 x}{7} + \frac{\cos^9 x}{9} + c$

Question 2

(a)(i) $3z + iw = 3(2+i) + i(1-i)$

$= 6 + 3i + i - i^2$

$= 7 + 4i$

(a)(ii) $z\overline{w} = (2+i)(1+i)$

$= 2 + 2i + i + i^2$

$= 1 + 3i$

(a)(iii) $\frac{5}{z} = \frac{5}{2+i} \times \frac{2-i}{2-i}$

$= \frac{5(2-i)}{4+1}$

$= 2 - i$

Q.2

(b)(i) Angle α is in the 2^{nd} quadrant with $\arg\alpha = \frac{5\pi}{6}$ and

$$(|\alpha|)^2 = (\sqrt{3})^2 + 1^2 \qquad \tan\alpha = -\frac{1}{\sqrt{3}}$$

$$= 3 + 1$$

$$|\alpha| = 2$$

Modulus-argument form $2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$

(b)(ii) De Moivre's theorem

$$\alpha^4 = (2\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})^4$$

$$= 2^4(\cos 4\times\frac{5\pi}{6} + i\sin 4\times\frac{5\pi}{6})$$

$$= 16(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3})$$

$$= 16(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$$

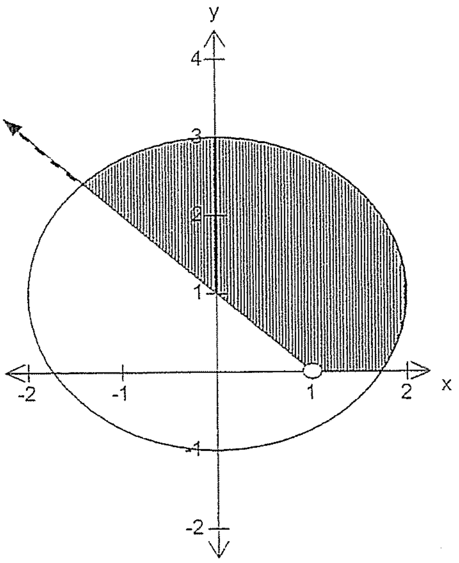
(b)(iii)

$$\alpha^4 = 16(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$$

$$= 16(-\frac{1}{2} + i \times -\frac{\sqrt{3}}{2})$$

$$= -8 - 8\sqrt{3}i$$

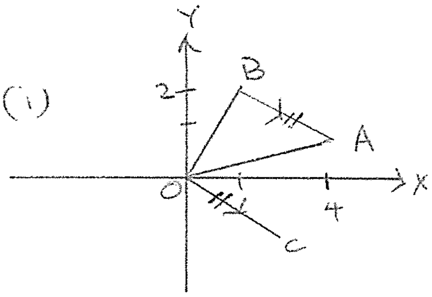
(c)



$|z - i| \le 2$ represents a region with a centre is (0, 1) and radius is less than or equal to 2.

$0 \le \arg(z-1) \le \frac{3\pi}{4}$ represents a region between angle 0 and $\frac{3\pi}{4}$ whose vertex is (1, 0), not including the vertex

2(d)(i)



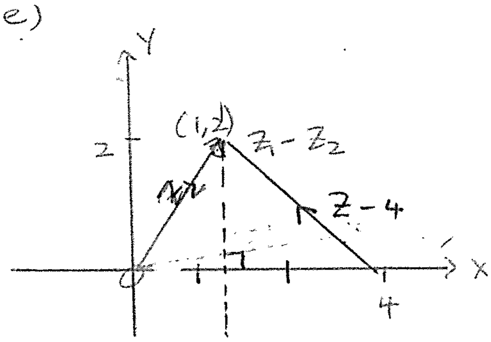
(ii) D rep

$$z_2 + i(z_1 - z_2)$$

$$= 1 + 2i + i(3 - i)$$

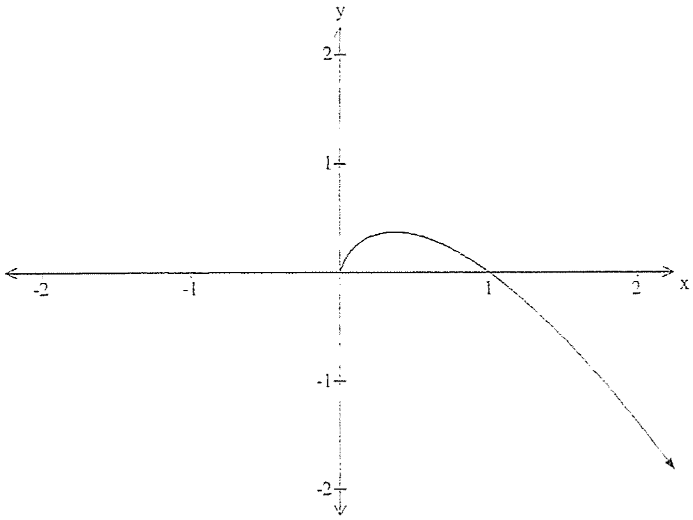
$$= 2 + 5i$$

2e)

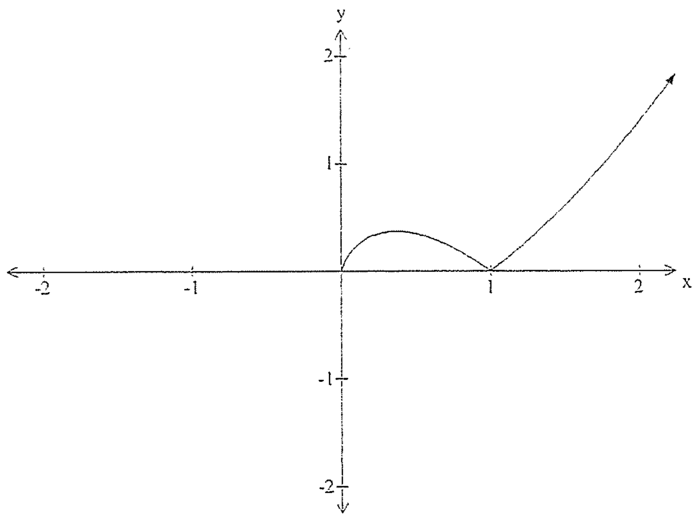


Question 3

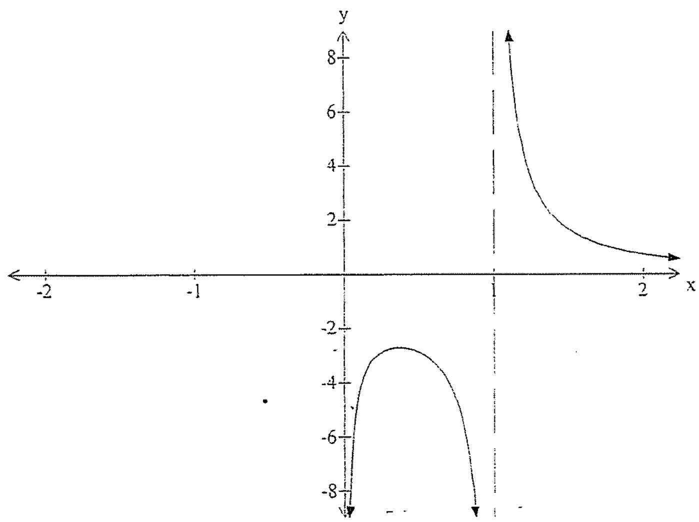
a)(i)



a)(ii)

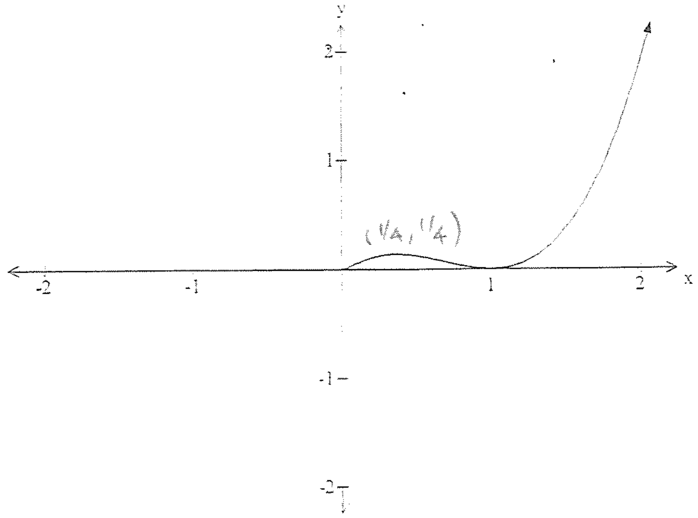


a)(iii)

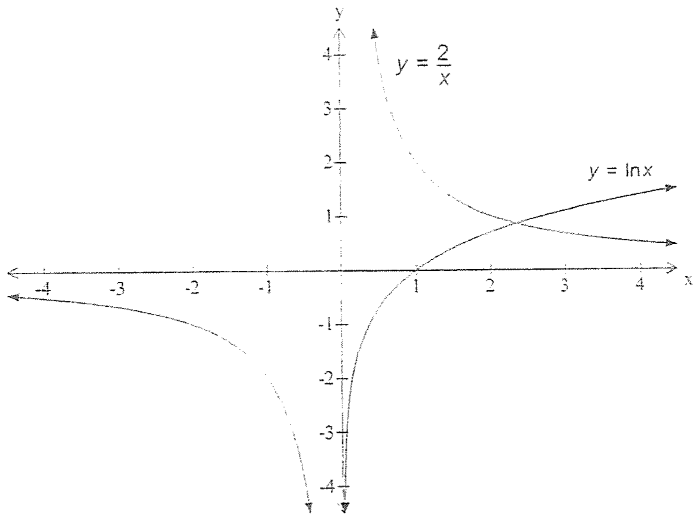


4

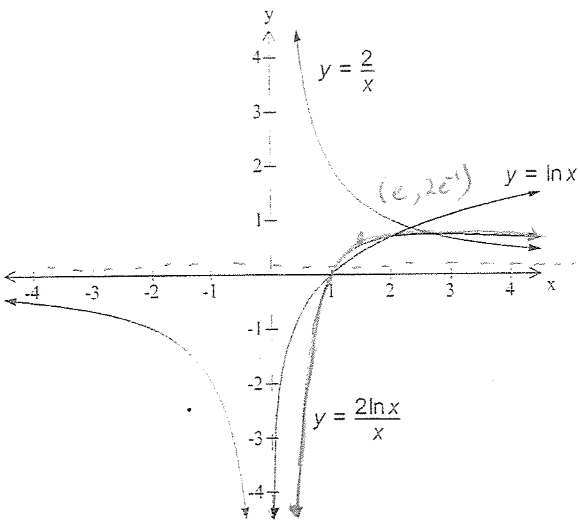
(a)(iv)


y
2
1
(1/4, 1/4)
-2
-1
1
2
x
-1
-2

(b)(i)

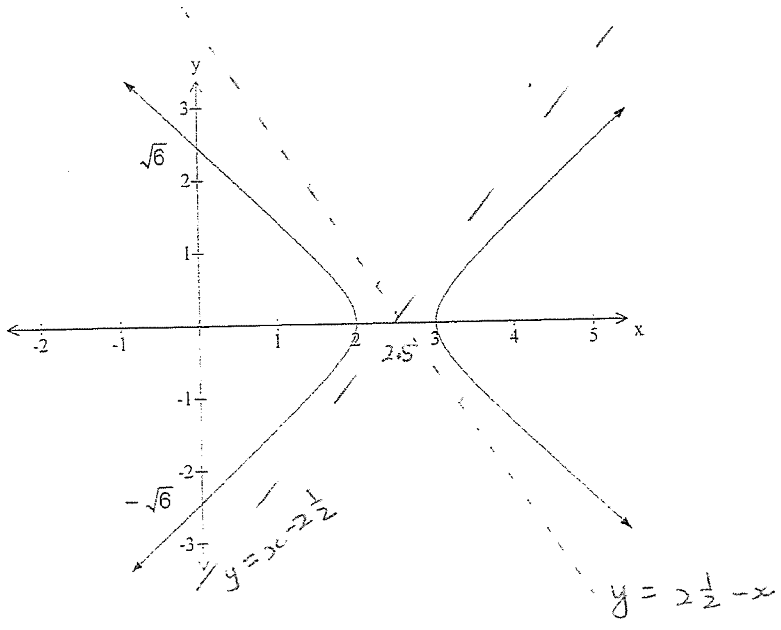

y
$y = \frac{2}{x}$
$y = \ln x$
x

(b)(ii)


y
$y = \frac{2}{x}$
$(e, 2e^{-1})$
$y = \ln x$
x
$y = \frac{2 \ln x}{x}$

Q3

c)



Question 4

(a)(i) Hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ has $a^2=16$ and $b^2=9$.

$a=4$ $\quad b=3$

Eccentricity

$$b^2=a^2(e^2-1)$$
$$9=16(e^2-1)$$
$$(e^2-1)=\frac{9}{16}$$
$$e^2=\frac{25}{16}$$
$$e=\frac{5}{4}$$

(a)(ii) Foci of a hyperbola are $(\pm ae,0)$.

Foci are $(\pm 5,0)$

X-intercepts of a hyperbola are ± 4.

The points are $(\pm 4,0)$

(a)(iii) Directrices of a hyperbola are $x=\pm\frac{a}{e}$.

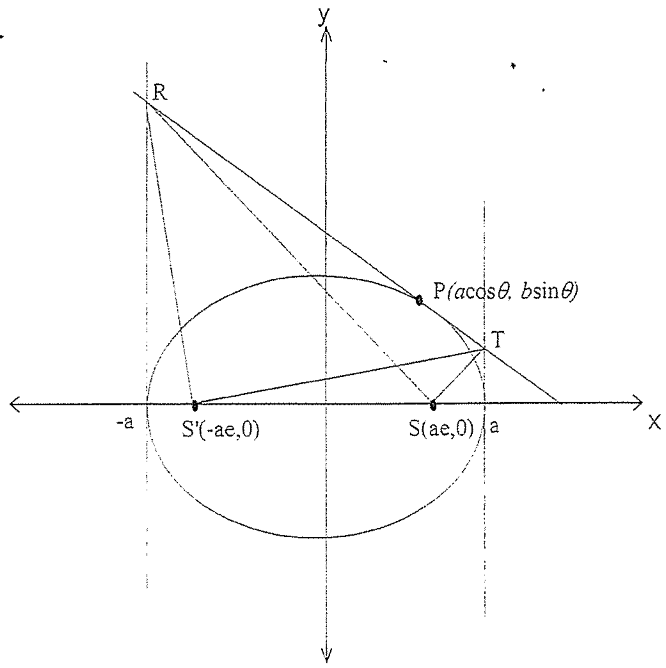
Directrices are $x=\pm\frac{16}{5}$

Asymptotes of a hyperbola are $y=\pm\frac{b}{a}x$.

Asymptotes are $y=\pm\frac{3}{4}x$

6

b) Q4



b)(i) To find the equation of tangent through P

$x = a\cos\theta \qquad y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= b\cos\theta \times \frac{1}{-a\sin\theta}$$

$$= \frac{-b\cos\theta}{a\sin\theta}$$

Equation of the tangent

$$y - y_1 = m(x - x_1)$$

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$$

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

(b)(ii) Q4

At T $x = a$ then $\frac{a}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$

$$\frac{y}{b}\sin\theta = 1 - \cos\theta$$

$$y = \frac{b(1-\cos\theta)}{\sin\theta}$$

At R $x = -a$ then similarly $y = \frac{b(1+\cos\theta)}{\sin\theta}$

Gradients of lines at the focus S(ae,0)

Gradient RS × Gradient TS

$$= \frac{\frac{b(1+\cos\theta)}{\sin\theta} - 0}{-a - ae} \times \frac{\frac{b(1-\cos\theta)}{\sin\theta} - 0}{a - ae}$$

$$= \frac{b(1+\cos\theta)}{-a(1+e)\sin\theta} \times \frac{b(1-\cos\theta)}{a(1-e)\sin\theta}$$

$$= \frac{b^2(1-\cos^2\theta)}{-a^2(1-e^2)\sin^2\theta}$$

$= -1$ $\quad \therefore \angle RST = 90^\circ$

(iii) If the diagram is reflected in the y-axis, the similar result follows for S'.

∴ R, T, S, S' are concyclic (opp ∠s suppl).

OR

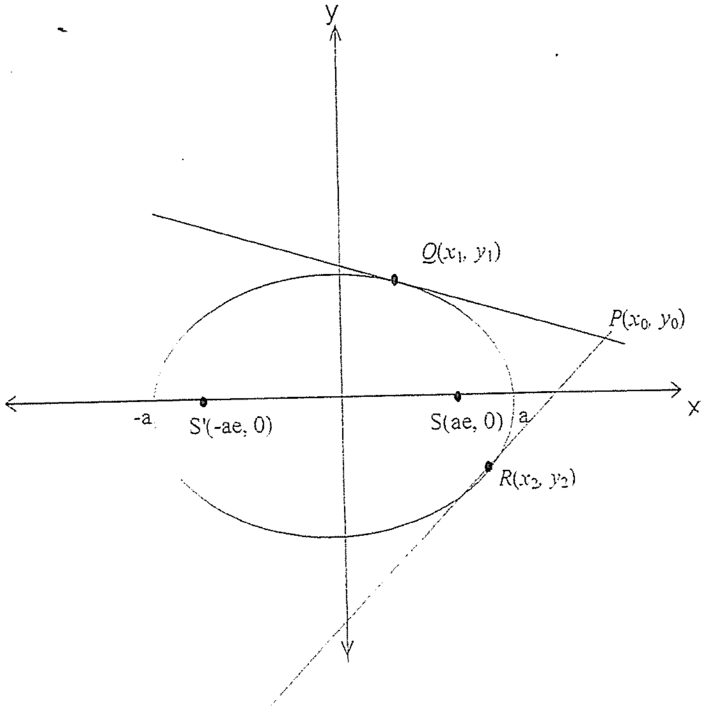
$$M_{TS} \times M_{RS} = -\frac{b^2}{a^2}\frac{(1-\cos^2\theta)}{(1-e^2)\sin^2\theta}$$

Change $S \to S'$ means change a^2 to $(-a)^2 = a^2$

Hence $m_{TS} \times m_{RS} = m_{TS'} \times m_{RS'} = -1$

∴ As $\angle RS'T = \angle RST$, then RTSS' are concyclic as by result ("angles subtended on same side of RT are equal")

) 4c)



To find the gradient of the tangent

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-2x}{a^2}\times\frac{b^2}{2y}$$

$$=\frac{-xb^2}{ya^2}$$

Equation of the tangent at $Q(x_1, y_1)$

$$y-y_1=m(x-x_1)$$

$$y-y_1=\frac{-x_1b^2}{y_1a^2}(x-x_1)$$

$$yy_1a^2-y_1^2a^2=-x_1xb^2+x_1^2b^2$$

$$x_1xb^2+yy_1a^2=y_1^2a^2+x_1^2b^2$$

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=\frac{y_1^2}{b^2}+\frac{x_1^2}{a^2}$$

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$$

$P(x_0, y_0)$ is on the tangent at Q.

$$\therefore \frac{x_1x_0}{a^2}+\frac{y_1y_0}{b^2}=1 \qquad \text{Eqn 1}$$

Similarly the equation of the tangent at $R(x_2, y_2)$ is

$$\frac{x_2x}{a^2}+\frac{y_2y}{b^2}=1$$

$P(x_0, y_0)$ is on the tangent at R

$$\therefore \frac{x_2x_0}{a^2}+\frac{y_2y_0}{b^2}=1 \qquad \text{Eqn 2}$$

Hence the points P & Q both satisfy

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$$

$\therefore$ The equation of the chord of contact PQ is $\frac{x_0x}{a^2}+\frac{y_0y}{b^2}=1$

9

Question 5

$a)$ $P(x) = x^4 + ax^2 + bx$

$x^2 + 1 = (x+i)(x-i)$

$P(i) \ = i^4 - a + bi$

but $P(i) = i + 2$ From $P(x) = (x^2+1)q(x) + x + 2$

$\therefore 1 - a = 2$ $bi = i$

$a = -1,$ $b = 1$

$b)$ $y = 2x^3 - 3x^2 - 12x + k$

need $y(2) \times y(-1) < 0$

$(2\times 8 - 3\times 4 - 24 + k)(-2 - 3 + 12 + k) < 0$

$(k-20)(k+7) < 0$

$-7 < k < 20$

$ci)$ 4!students and 3! teachers, $\therefore P_1P_2 = 2\times 3!\times 4! = 288$

$cii)$ S_1P_1 represents student, parent. S', P are remaining set of students and other parents, T is teachers.

$\therefore 3! =$ students and $3! =$ teachers.

$2\times 3!\times 3! = 72$

$\therefore \text{Prob} = \dfrac{72}{288} = \dfrac{1}{4}$

$5di)\ \alpha+\beta+\gamma=\frac{-b}{a} \qquad \alpha\beta+\beta\gamma+\alpha\gamma=\frac{c}{a} \qquad \alpha\beta\gamma=\frac{-d}{a}$

$dii)\ \alpha+\beta+\gamma=2\alpha=\frac{12}{36}=\frac{1}{3},\ \ \alpha=\frac{1}{6}$

$$diii)\ \alpha+\beta+\gamma=2(\beta+\gamma)=-p$$

$$\begin{aligned}\alpha\beta+\beta\gamma+\alpha\gamma &= \beta\gamma+(\beta+\gamma)\alpha \\ &= \beta\gamma+(\beta+\gamma)^2 \\ &= (\beta^2+3\beta\gamma+\gamma^2) \\ &= q\end{aligned}$$

$$\alpha\beta\gamma=\beta\gamma+(\beta+\gamma)=-r$$

$$\begin{aligned}&\therefore p^3-4pq+8r \\ &=-8(\beta+\gamma)^3+8(\beta+\gamma)+(\beta^2+3\beta\gamma+\gamma^2)-8(\beta+\gamma)\beta\gamma \\ &=-8(\beta+\gamma)\left[(\beta+\gamma)^2-(\beta^2+3\beta\gamma+\gamma^2)+\beta\gamma\right] \\ &=0\end{aligned}$$

Question 6

Assume $AB = x$

$ai)$ join C to $E, \angle AEC = 90^o$

CE is the axis of symmetry for ΔABC.

$$EB = \frac{x}{2}$$

$$\therefore \cos\alpha = \frac{EB}{BC} = \frac{x/2}{y} = \frac{x}{2y}.$$

$ii)$ join A to $D, \angle ADC = 90^o$

$$\cos\alpha = \frac{BD}{AB}$$

$$\therefore BD = AB\cos\alpha$$

$$= x.\frac{x}{2y} = \frac{x^2}{2y}$$

$$\therefore DC = y - \frac{x^2}{2y}$$

$$= \frac{2y^2 - x^2}{2y}$$

$$bi) I_n = \int_0^{\frac{\pi}{2}} \cos^n t\, dt$$

$$= \left[\sin t.\cos^{n-1} t\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \sin^2 t.\cos^{n-2} t\, dt$$

$$I_n = (n-1)\int_0^{\frac{\pi}{2}} (1-\cos^2 t).\cos^{n-2} t\, dt$$

$$= (n-1)[I_{n-2} - I_n]$$

$$= (n-1)I_{n-2} - nI_n + I_n$$

$$nI_n = (n-1)I_{n-2}$$

$$I_n = \frac{(n-1)}{n} I_{n-2}, \text{ for } n \ge 2$$

$$bii) I_4 = \frac{3}{4} I_2$$

$$= \frac{3.1}{4.2} I_0$$

$$\int_0^{\frac{\pi}{2}} dt = [t]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2}$$

$$I_4 = \frac{3}{8} + \frac{\pi}{2}$$

$$= \frac{3\pi}{16}$$

6ci) $0 < \dfrac{(k+1)^{k-1}}{k^k} < 1$

so $\quad (k+1)^{k+1} > (k+1)^{k+1} \times \dfrac{(k+1)^{k-1}}{k^k}$

$\therefore \quad (k+1)^{k+1} > \dfrac{(k+1)^{(k+1+k-1)}}{k^k} = \dfrac{(k+1)^{2k}}{k^k}$

6cii) Let $P(n)$ be the proposition that

$n^n > (n+1)^{n-1}$ for $n \geq 2$

$P(2): 2^2 > (2+1)^{2-1}$,

$LHS = 4, \qquad RHS = 3$

$\therefore$

$LHS > RHS$, so true for $P(2)$

To Prove : If $P(k)$ is true then $P(k+1)$ is true for $k \geq 2$.

$P(k):\ k^k > (k+1)^{k-1}$

$P(k+1): (k+1)^{k+1} > (k+2)^k$.

Consider LHS of $P(k+1)$

Assuming $P(k)$ true then $k^k > (k+1)^{k-1}$,

$k > 0$, then $k^k > (k+1)^{k-1}$

$$\frac{(k+1)^{k-1}}{k^k} < 1$$

So by i) $\quad (k+1)^{k+1} > \dfrac{(k+1)^{2k}}{k^k}$

$$= \left[\frac{(k+1)^2}{k}\right]^k$$

$$= \left(\frac{k^2+2k+1}{k}\right)^k$$

$$= \left(k+2+\frac{1}{k}\right)^k$$

$> (k+2)^k$ as $\quad k > 0$

$\therefore P(k+1)$ true provided $P(k)$ true. Hence $P(n)$ true for integeers $n \geq 2$, by Mathematical Induction.

$$7ai)\ I_n = \int \frac{dx}{\left(x^2+1\right)^n},\quad n \geq 1$$

$$= \int \frac{1}{\left(x^2+1\right)^n} \frac{d}{dx}(x)\, dx$$

$$= \frac{x}{\left(x^2+1\right)^n} - (-n)\int x.\frac{2x}{\left(x^2+1\right)^{n+1}}\, dx,\quad n \geq 1$$

$$= \frac{x}{\left(x^2+1\right)^n} + 2n\int \frac{x^2}{\left(x^2+1\right)^{n+1}}$$

$$= \frac{x}{\left(x^2+1\right)^n} + 2n\int \left[\frac{x^2+1}{\left(x^2+1\right)^{n+1}} - \frac{1}{\left(x^2+1\right)^{n+1}}\right] dx$$

$$= \frac{x}{\left(x^2+1\right)^n} + 2n\int \frac{dx}{\left(x^2+1\right)^n} - 2n\int \frac{dx}{\left(x^2+1\right)^{n+1}}$$

$$\therefore I_n = \frac{x}{\left(x^2+1\right)^n} + 2n\, I_n - 2n\, I_{n+1}$$

$$\therefore 2n\, I_{n+1} = \frac{x}{\left(x^2+1\right)^n} + (2n-1)\, I_n$$

$$\therefore I_{n+1} = \frac{1}{2n}\left[\frac{x}{\left(x^2+1\right)^n} + (2n-1)\, I_n\right]$$

replace n by n-1, for $n \geq 2$

$$I_n = \frac{1}{2(n-1)}\left[\frac{x}{\left(x^2+1\right)^{n-1}} + (2n-3)\, I_{n-1}\right]$$

Q7

7aii) $I_n = \dfrac{1}{2(n-1)}\left\{\left[\dfrac{x}{(x^2+1)^{n-1}}\right]_0^1 + (2n-3)I_{n-1}\right\}$

$= \dfrac{1}{2(n-1)}\left[\dfrac{1}{2} + (2n-3)I_{n-1}\right], n \geq 2$

$I_2 = \int_0^1 \dfrac{dx}{(x^2+1)^2}$

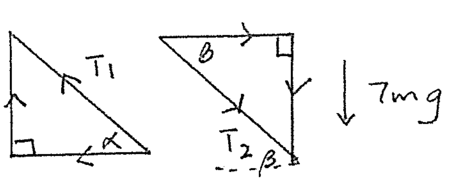
$= \dfrac{1}{2}\left[\dfrac{1}{2} + I_1\right]$

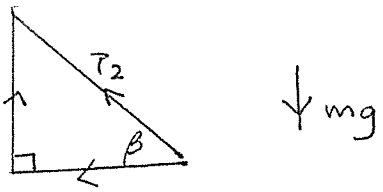
$I_1 = \int_0^1 \dfrac{dx}{(x^2+1)} = \left[\tan^{-1} x\right]_0^1$

$= \tan^{-1} 1 - 0 = \dfrac{\pi}{4}$

$\therefore I_2 = \dfrac{1}{2}\left[\dfrac{1}{2} + \dfrac{\pi}{4}\right] = \dfrac{1}{4} + \dfrac{\pi}{8}$

7b)





$T_1 \cos\alpha - T_2 \cos\beta = 7maw^2$ (1)

$T_1 \sin\alpha = T_2 \sin\beta + 7mg$ (2)

$\cos\alpha = \dfrac{a}{5a/3} = \dfrac{3}{5}$

$\sin\alpha = \dfrac{4}{5}$

$T_2 \cos\beta = m.2a.w^2$ (3)

$T_2 \sin\beta = mg$ (4)

$\cos\beta = \dfrac{a}{5a/4} = \dfrac{4}{5}$

$\sin\beta = \dfrac{3}{5}$

$7bi)\quad T_2 = \frac{mg}{\sin\beta} = mg \times \frac{5}{3} = \frac{5}{3}mg$

$bii)$ by(2)

$$T_1 \times \frac{4}{5} = \frac{5}{3}mg \times \frac{3}{5} + 7mg$$

$$\frac{4}{5}T_1 = 8mg$$

$$\therefore T_1 = \frac{5}{4}8mg = 10mg$$

$iii)$ by(1)

$$T_1 \cos\alpha - T_2 \cos\beta = \frac{7mv_1^2}{a}$$

$$10mg \times \frac{3}{5} - \frac{5}{3}mg \times \frac{4}{5} = \frac{7mv_1^2}{a}$$

$$\therefore v_1^2 = \frac{a}{7m}\left[10mg \times \frac{3}{5} - \frac{5}{3}mg \times \frac{4}{5}\right]$$

$$= \frac{ag}{7}\left[6 - \frac{4}{3}\right]$$

$$= \frac{ag}{7} \times \frac{14}{3} = \frac{2}{3}ag$$

also by (4)

$$T_2 \cos\beta = \frac{mv_2^2}{2a}$$

$$\frac{5}{3}mg \times \frac{4}{5} = \frac{m}{2a}v_2^2$$

$$v_2^2 = \frac{2a}{m} \times \frac{5}{3}mg \times \frac{4}{5}$$

$$= ag\left[2 \times \frac{5}{3} \times \frac{4}{5}\right]$$

$$= \frac{8}{3}ag$$

$$\therefore \left(\frac{v_1}{v_2}\right)^2 = \frac{2/3}{8/3} == \frac{1}{4}$$

$$\left(\frac{v_1}{v_2}\right) = \frac{1}{2}$$

Question 8

$a)$ Area of slice $= \pi(1-y)^2$ $\qquad$ Vol slice $= \pi(1-\sin x)^2 \, \Delta x$

$$= \pi(1-\sin x)^2$$

$$V = \lim_{\Delta x \to 0} \sum_{0}^{\frac{\pi}{2}} \pi(1-\sin x)^2 \, dx$$

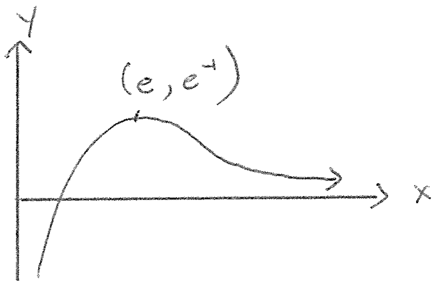
$$= \pi \int_0^{\frac{\pi}{2}} 1 - 2\sin x + \frac{1}{2} - \frac{1}{2}\cos 2x \, dx \qquad \left(\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x \, dx \right)$$

$$= \pi \int_0^{\frac{\pi}{2}} \left(\frac{3}{2} - 2\sin x - \frac{1}{2}\cos 2x \right) dx$$

$$= \pi \left[\frac{3}{2}x + 2\cos x - \frac{1}{4}\sin 2x \right]_0^{\frac{\pi}{2}}$$

$$= \pi \left(\frac{3\pi}{4} - 2 \right) u^3$$

$bi)$ $\quad h(x) = \dfrac{\log_e x}{x}$



$$h'(x) = \frac{\frac{1}{x} \times x - \ln x}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

$bii)$ St.Pt. is at $x = e$, $y = e^{-1}$

assuming $\pi > e$, then from the graph

$$\frac{\ln \pi}{\pi} < e^{-1} = \frac{1}{e}$$

$$\therefore \ln \pi < \frac{\pi}{e}$$

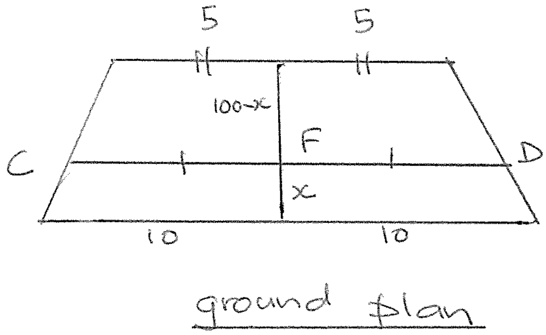
$$\therefore e^{\ln \pi} < e^{\frac{\pi}{e}}$$

$$\left(e^{\ln \pi}\right)^e < \left(e^{\frac{\pi}{e}}\right)^e$$

$$\pi^e < e^\pi$$

$$ie \quad e^\pi > \pi^e$$

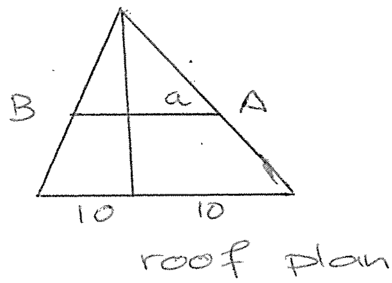
8ci) $\dfrac{FD}{200-x} = \dfrac{10}{200}$



$FD = \dfrac{200-x}{20} = 10 - \dfrac{x}{20}$

$\therefore DC = \left(20 - \dfrac{x}{10}\right)$ as reqd.

cii) $\dfrac{a}{100-x} = \dfrac{10}{100}$

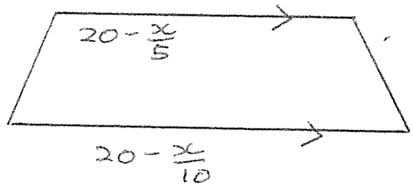


$a = \dfrac{100-x}{10}$

$a = 10 - \dfrac{x}{10}$

$\therefore AB = 2a = \left(20 - \dfrac{x}{5}\right)$

each trap slice looks like:



Area of face is:

$A_1 = 10\left(20 - \dfrac{x}{5} + 20 - \dfrac{x}{20}\right)$

$A_1 = 400 - 3x$

Vol of each slice is ΔV where $\Delta V = (400-3x)\Delta x$

$\therefore V = \lim\limits_{\Delta x \to 0} \sum (400-3x)\Delta x$

$= \displaystyle\int_0^{100} (400-3x)\,dx$

$= \left[400x - \dfrac{3x^2}{2}\right]_0^{100}$

$= 25000\ m^3$

End of solutions

Marking Guidelines for THGS

Mathematics Extension 2 Trial HSC 2011

Preamble

- These guidelines were strictly implemented. Very occasionally a 'Judgment Call' may have been made. Marks were awarded, not deducted, for student's working as per guidelines.
- The 'carry through error policy' was used unless the solution was made easier by the student's error. In these cases, a 'Judgment Call' was made.
- Multiple Solutions policy. The last attempt (not crossed out) is assumed to be the actual attempt, whether or not other attempts have been crossed out. If a crossed out attempt would have gained more marks, one mark was deducted, and if higher then this became the mark.
- A red line down the left or right hand side of a script page means the content has been read by the marker.
- For each question part, only the total mark is given on the right hand side of the page, usually without comment. Sometimes working is underlined in red usually indicating that a problem with the student's working is noted.

Question 1

1(a) **2 marks**

- Correct bald answer-ignore '+C'.
- Correct solution-ignore '+C'.

1 mark

- Correct integral from a correct substitution.
- Answer of the form $k\sqrt{16-x^2}$ obtained from a correct integral.

1(b) **2 marks**

- Correct bald answer-ignore '+C'.
- Correct solution-ignore '+C'.

1 mark

- Correct integral with the square completed.
- Obtains answer of the form $k\tan^{-1}(\frac{x+2}{a})$ obtained from their integral.

1(c)(i) <u>**3 marks**</u>

- Correct answer or solution.

<u>**2 marks**</u>

- Obtains a set of answers by using a correct method.

<u>**1 mark**</u>

- Obtains a correct equation in *a, b, c*.

1(c)(ii) <u>**2 marks**</u>

- Correct answer or solution.
- Obtains a correct answer from their set of values of *a, b , c.* (Ignore '+C'.)

<u>**1 mark**</u>

- Obtains a set of three integrals using their set of values of *a,b,c*.
- Obtains an answer of the form $k\ln(x^2+1)+l\tan^{-1}x+m\ln(x+2)$.

1(d) <u>**3 marks**</u>

- Correct solution-ignore '+C'.

<u>**2 marks**</u>

- Obtains an answer of the form $kx^3\ln(x)-mx^3$,with *m>0*, using a correct method.

<u>**1 mark**</u>

- Attempts to use correctly the method of 'Integration by Parts'.

1(e) <u>**3 marks**</u>

- Correct solution.

<u>**2 marks**</u>

- Obtains a correct integral in terms of a polynomial in *u*.
- Obtains the correct answer, in terms of $\cos(x)$, from their integrand expressed as a polynomial in *u*.

<u>**1 mark**</u>

- Correctly substitutes $u=\cos(x)$.
- Obtains a correct answer, in terms of *u*, from their integrand.

Question 2

2(a)(i), (ii), (ii)

1 mark each-correct answer.

2(b)(i)

2 marks
- Correct answer.

1 mark
- Correct modulus.
- A correct argument.

2(b)(ii)

1 mark
- A correct answer.
- A correct answer from their answer to 2(b)(i).

2(b)(iii)

1 mark
- A correct answer.
- A correct answer from their answer to 2(b)(ii).

2(c)

3 marks
- Correct diagram.(Ignore no open circle on real axis at 1, and, heavy line on all of the ray $\arg(z-1)=\frac{3\pi}{4}$).

2 marks
- Diagram showing loci $|z-i|=2$ and $\arg(z-1)=\frac{3\pi}{4}$ correctly.
- Diagram with one of the two loci shown correctly with correct shading for their diagram.

1 mark
- Shading correct for their diagram with their two loci.
- One of the two loci drawn correctly.

2(d)(i)

1 mark-correct diagram with AB parallel to OC and AB=OC marked or indicated in some way.

2(d)(ii)

1 mark-correct answer.

2(e)

3 marks

- Correct sketch and geometrical description of locus.

2 marks

- Correct equation for locus.
- Correct sketch of locus.
- Correct geometrical description of locus.

1 mark

- Attempts to find equation of locus by a correct method.

Question 3

3(a)(i) **1 mark**-Correct graph shape.

3(a)(ii) **1 mark**-Correct graph shape including cusp on *x*-axis.

3(a)(iii)

2marks

- Correct graph shape including the two vertical and the horizontal asymptotes.

1 mark

- Correct graph shape.
- The asymptotes $x=1$ and $y=0$ shown correctly on diagram.

3(a)(iv)

2 marks

- Correct graph shape including the two turning points (NOTE: gradient is indeterminate as $x \to 0^+$ from the given information, so shape ignored at $x=0$).

1 mark

- Graph is shown to be non-negative.

3(b)(i) **1 mark** – Correct graph shapes of $y=\frac{2}{x}$ and $y=\ln(x)$.

3(b)(ii)

4 marks

- Correct graph showing correct co-ordinates of turning point, *x*-intercept and indication of the asymptotes.

3 marks

- Correct graph shape shown with correct co-ordinates of turning point determined.
- Correct graph shape shown with the correct two asymptotes indicated and the *x*-intercept given.

2 marks

- Co-ordinate of the turning point correctly determined.
- Correct two asymptotes and *x*-intercept indicated.

1 mark

- The horizontal asymptote is correctly determined.
- Correct graph shape is shown.

3(c)

4 marks

- Correct graph shape shown with the following correctly indicated: the two *x*-intercepts, the two *y*-intercepts, the two asymptotes, with respective equations and common *x*-intercept.

3 marks

- Correct graph shape shown with the two *x*-intercepts and *y*-intercepts correctly shown.
- Correct graph shape shown with the two *x*-intercepts and the two asymptotes (no equations given, but common *x*-intercept given).

2 marks

- Correct graph shape shown with correct *x*-intercepts.

1 mark

- Correct graph shape shown.
- Asymptotes correctly shown (no equation or *x*-intercept given)
- Correct two *x*-intercepts and *y*-intercepts shown.
- Graph displays *x*-axis as an axis of symmetry.

Question 4

4(a)(i) **1 mark**- Correct answer.

4(a)(ii)

2 marks
- Correct answers, or, correct answers using their value of *e* from 4(a)(i).

1 marks
- Obtains correct co-ordinates of foci, or, from using their value of *e* in 4(a)(i).
- Obtains correct *x*-intercepts.

4(a)(iii)

2 marks
- Correct answers, or, correct answers using their value of *e* in 4(a)(i).

1mark
- Obtains correct directrices, or, correct from using their value of *e* in 4(a)(i).
- Obtains correct asymptotes, or, correct from using their value of *e* in 4(a)(i).

4(b)(i)

2 marks
- Correct solution-must show the use of $\sin^2\theta+\cos^2\theta=1$.

1 mark
- Obtains correct gradient formula in terms of θ.
- Uses $\sin^2\theta+\cos^2\theta=1$ to achieve their answer.

4(b)(ii)

3 marks
- A correct solution.

2 marks
- Obtains correct expressions for the gradients of *RS* and *TS*.

1 mark
- Attempts to use $m_1m_2=-1$ in their solution.

4(b)(iii) **1 mark**-uses a correct method of proof.

4(c)

4 marks

- A correct solution.

3 marks

- Obtains the Cartesian equation of tangent to the ellipse and attempts to use it to find the chord of contact by a correct method.

2 marks

- Obtains Cartesian equation of tangent to the ellipse by a correct method.
- Uses a correct reasoning to find the chord of contact.

1 mark

- Obtains gradient formula for tangent at $P(x_1, y_1)$ or similar point on ellipse.
- Uses $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$ or similar result in finding equation of tangent to ellipse.
- Attempts to use correct reasoning in finding the equation of the chord of contact.

Question 5

5(a)

3 marks

- Correct solution.

2marks

- Obtains a correct equation in *a, b*.
- Obtains correct values of *a, b* from their equation(s).

1 mark

- Obtains $x^2+1=(x+i)(x-i)$.
- Obtains $P(x)=(x^2+1)Q(x)+(x+2)$ or similar.
- Obtains one of $P(\pm i)=\pm i+2$ or $1-a\pm ib$.

5(b)

2 marks

- Correct solution.

1 mark

- Obtains a correct possible graph of $y=y(x)$.
- Obtains a correct condition(s) for three real and distinct roots.

5(c)(i)

2 marks
- Correct answer in any form.

1 mark
- Obtains answer of $3!\times4!$ or equivalent merit.
- Shows correct reasoning for permutations in a circle.

5(c)(ii)

2 marks
- Correct answer in any form.
- Correct solution using their reasoning and answer from 5(c)(i).

1mark
- Obtains $2\times3!\times3!$ as the number of arrangements.
- Shows correct reasoning for their circle permutation formula used in 5(c)(i).

5(d)(i) **1 mark** – Correct answer.

5(d)(ii)

2 marks
- Correct answer.

1 mark
- Obtains $\alpha+\beta+\gamma=2\alpha$.
- Obtains $\alpha+\beta+\gamma=\frac{1}{3}$.

5(d)(iii)

3 marks
- Correct solution.

2 marks
- Obtains *p,q,r* in terms of β,γ .
- Obtains $\alpha=-\frac{p}{2}$ as a root and attempts to substitute into the equation.

1 mark
- Correctly finds one of the sum of roots, sum of roots taken two at a time, product of roots.
- Correctly finds $-\frac{p}{2}$ as a root of the equation.

Question 6

6(a)(i)

2 marks

- A correct solution.

1 mark

- Showing *EC* is perpendicular to *AB*, or equivalent merit.
- Showing $EB = AE = \frac{x}{2}$.
- Attempts to use the cosine rule.

6(a)(ii)

4 marks

- A correct solution.

3 marks

- Obtains correctly *BD* in terms of *x,y*.
- Obtains correctly $\cos(2\alpha)$ in terms of *x,y*.

2 marks

- Attempts to find *BD* by a correct method.
- Attempts to find *DC* by a correct method.

1 mark

- Shows *AD* is perpendicular to *BC*.
- Shows $\angle ACB = \pi - 2\alpha$.

6(b)(i)

2 marks

- A correct solution.

1 mark

- Shows $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^2(t)\cos^{n-2}(t)dt$.
- Shows $I_n = I_{n-2} - \frac{1}{n-1} I_n$ or equivalent merit.

6(b)(ii)

2 marks

- A correct solution.

1 mark

- Obtains $I_0 = \frac{\pi}{2}$.
- Uses the reduction formula correctly.

6(c)(i)

2 marks

- A correct solution.

1 mark

- Obtains that $(k+1)^{k+1} > (k+1)^{k+1} \times \frac{(k+1)^{k-1}}{k^k}$.
- Obtains $\frac{(k+1)^{2k}}{k^k} = (k+1)^{k+1} \times \frac{(k+1)^{k-1}}{k^k}$.

6(c)(ii)

3 marks

- A correct proof.

2 marks

- Shows statement true for $n = 2$ and uses 6(c)(i) in the inductive step or equivalent merit.

1 mark

- Shows statement true for $n = 2$.

Question 7

7(a)(i)

4 marks
- A correct solution.

3 marks
- Obtains I_n in terms of I_{n-1} or I_{n+1}.

2 marks
- Correctly expresses $\dfrac{x^2}{(x^2+1)^{n+1}}$ or $\dfrac{1}{(x^2+1)^n}$ as two partial fractions.

1 mark
- Attempts to use integration by parts.

7(a)(ii)

2 marks
- A correct solution.

1 mark
- Correctly uses the reduction formula for the given integral.
- Correctly obtains $\displaystyle\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$.

7(b) *There was a typographical error in this question. The stem was correctly worded but the diagram had mass B marked incorrectly as* ***m*** *instead of correctly as* ***7m.*** *The marking took this into account so that no student was disadvantaged. This would not have affected the outcome to 7(b)(i) .*

7(b)(i)

3 marks
- A correct solution.

2 marks
- Obtains $T_2 \sin(\beta) = mg$.

1 mark
- Obtains $\sin(\beta) = \dfrac{3}{5}$.
- Obtains a correct vector diagram for T_2.

7(b)(ii)

2 marks

- A correct solution using mass at *B* either 7*m* or *m*.

1 mark

- Correctly obtains $T_1\sin(\alpha) = T_2\sin(\beta) + \begin{cases} 7mg \\ mg \end{cases}$ or equivalent.
- Correctly obtains T_1 from their equations.

7(b)(iii)

4 marks

- A correct solution.
- A correct solution using their results from 7(b)(i) and (ii).

3 marks

- Correctly finds v_1^2 or v_2^2 by using a correct method.

2 marks

- Correctly obtains the equations $T_1\cos(\alpha) - T_2\cos(\beta) = \begin{cases} \dfrac{7mv_1^2}{a} \\ \dfrac{mv_1^2}{a} \end{cases}$ and $T_2\cos(\beta) = \dfrac{mv_2^2}{2a}$ or equivalent.
- Correctly obtains $\dfrac{v_1}{v_2}$ from their equations using their results from 7(b)(i) and (ii).

1 mark

- Shows vector force diagrams for particle *B*.
- Correctly obtains $T_1\cos(\alpha) - T_2\cos(\beta) = \begin{cases} \dfrac{7mv_1^2}{a} \\ \dfrac{mv_1^2}{a} \end{cases}$ or equivalent.
- Obtains $\cos(\beta) = \dfrac{4}{5}$.

Question 8

8(a)

4 marks

- A correct solution.

3 marks

- Obtains a correct primitive from a correct integral for the volume.
- Substitutes correctly into their primitive from a correct integral for the volume.

2 marks

- Obtains a correct integral for the volume.
- Obtains correctly a volume from their integral whose integrand involves a sine function.

1 mark

- Obtains a correct integrand in their integral for the volume.
- Obtains a correct expression for their 'slice' volume.
- Obtains a correct integral from their 'slice' volume.

8(b)(i) **1 mark** – A correct answer.

8(b)(ii)

2 marks

- A correct proof.

1 mark

- Obtains correctly $\frac{\ln(\pi)}{\pi} < e^{-1}$ or equivalent merit.
- Obtains, for *x>0*, that e^{-1} is the global maximum of *h(x)*.

8(c)(i)

2 marks

- A correct proof.

1 mark

- Obtains a correct statement using similar triangles related to the floor of the showroom.

8(c)(ii)

6 marks

- A correct solution.

5 marks

- Obtains a correct primitive from a correct integral for the volume.

4 marks

- Obtains a correct integral for the volume.
- Obtains a correct volume from their integral.

3 marks

- Obtains a correct integrand.
- Obtains a correct primitive from their integral for the volume.

2 marks

- Obtains a correct expression for the area of the trapezium *ABCD*.

1 mark

- Obtains a correct statement using similar triangles related to the ceiling of the showroom.

END OF MARKING GUIDELINES